

Palatini Variational Principle for N -Dimensional Dilaton Gravity

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Abstract

We consider a Palatini variation on a general N -Dimensional second order, torsion-free dilaton gravity action and determine the resulting equations of motion. Consistency is checked by considering the restraint imposed due to invariance of the matter action under simple coordinate transformations, and the special case of $N = 2$ is examined. We also examine a sub-class of theories whereby a Palatini variation dynamically coincides with that of the "ordinary" Hilbert variational principle; in particular we examine a generalized Brans-Dicke theory and the associated role of conformal transformations.

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1 Introduction

Dilaton theories of gravity are playing an increasingly important role in the study of gravitational physics. Such theories are a generalization of general relativity in which the basic field variables that describe gravity consist of a symmetric rank-2 tensor (the metric) and one (sometimes more) scalar field referred to as the dilaton. The prototype of this class of theories is Brans-Dicke theory [1], whose original motivation stemmed from developing a theory which incorporated Mach's principle by relating the gravitational constant G to the mean value of a scalar field which was coupled to the mass density of the universe [2]. This motivation has largely been transplanted by superstring theories [3], which generically predict that the low-energy effective Lagrangian governing gravitational dynamics is that of a dilaton theory of gravity.

The generic form for the gravitational action for such theories is

$$S = \int d^N x \sqrt{-g} [D(\Psi)R(g) + A(\Psi)(\nabla\Psi)^2 + 16\pi\mathcal{L}_m(\Phi, \Psi)] + S_B \quad (1)$$

where Ψ is the dilaton field, $g_{\mu\nu}$ is the metric, and Φ symbolically denotes the matter fields, whose Lagrangian \mathcal{L}_m may or may not also have explicit dependence on Ψ . The gravitational field equations for such an action are derived by extremizing it with respect to variations in the metric and dilaton fields. Because the first term in the action is of 2nd order in metric derivatives, it is necessary to add on the boundary term S_B in (1) so that the variational principle is well defined. In particular, both the variation of the induced metric and its derivatives must be held fixed on the boundary. Alternatively, the action may be supplemented with additional boundary terms such that we need only fix the induced metric on the boundary. Inclusion of such boundary terms is essential in order to correctly the thermodynamics of a system of matter fields coupled to dilatonic gravity [4].

However an alternate variational principle exists for gravitational theories in which the connection is elevated to the status of an independent gravitational field variable, on par with the metric and the dilaton (if any). Referred to as the Palatini variational principle, the action for N -dimensional general relativity takes the form

$$S_{EH}[g, \Gamma] = \int d^N x [\sqrt{-g} (R(\Gamma) + 16\pi\mathcal{L}_m)] \quad (2)$$

where

$$R(\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) = g^{\mu\nu} \left[\Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\mu\rho,\nu}^\rho + \Gamma_{\rho\eta}^\rho \Gamma_{\mu\nu}^\eta - \Gamma_{\nu\eta}^\rho \Gamma_{\mu\rho}^\eta \right] \quad (3)$$

is the Ricci scalar. In the Palatini approach the action (2) is subject to the independent variations $\delta_g S = 0$ and $\delta_\Gamma S = 0$, of the metric and the connection $\Gamma_{\mu\nu}^\alpha$ respectively. The former variation yields

$$8\pi T_{\mu\nu} = G_{\mu\nu}(\Gamma), \quad (4)$$

where $T_{\mu\nu}$ is the stress energy of the matter fields and $G_{\mu\nu}$ is the Einstein tensor of the manifold. Variation with respect to the connection yields

$$\partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\mu}^\eta g_{\eta\nu} - \Gamma_{\lambda\nu}^\eta g_{\mu\eta} = 0 \quad (5)$$

which is the condition of metric compatibility, whose solution

$$\Gamma_{\mu\nu}^\eta = \left\{ \begin{matrix} \eta \\ \mu \quad \nu \end{matrix} \right\} \quad (6)$$

is the Christoffel symbol. Hence the geometrical constraint (5) implicitly assumed in (1) arises as a field equation. A curious feature of the above approach is that there is no need to include a boundary term, since the field variations are assumed to vanish on the boundary [5].

We consider in this paper the field equations and resultant dynamics which arise from a Palatini variational principle applied to dilatonic gravitational theories. This “connection-oriented” perspective is in part motivated by a potential future quantization procedure anticipated by dilaton gravity theories, as well as by an interest in exploring the relationship between metric compatibility and extremization of the action. For N -dimensional general relativity we have demonstrated in a recent paper [6] that this relationship can be understood to arise as a consequence of the breaking of a maximal deformation symmetry [7] associated with a transformation of the connection variables. In this paper, however, we are solely concerned with how metricity arises (or not) explicitly resulting from the contributions of the dilaton sector of the generalized action. That is, we deliberately break the aforementioned deformation symmetry to isolate dilaton-induced effects and chose as our starting point a generalized dilaton action whose constraints are solely that it be

first-order in curvature terms, at most quadratic in derivatives of Ψ , and with a matter action only dependent on the metric (and hence independent of both the connection and the dilaton field).

The action we consider is thus of the form

$$S = \int d^N x \sqrt{-g} [D(\Psi)R(\Gamma) + A(\Psi)(\nabla\Psi)^2 + B(\Psi)(\nabla^\nu\Psi)g_{\alpha\beta}\nabla_\nu g^{\alpha\beta} + C(\Psi)(\nabla_\mu\Psi)(\nabla_\nu g^{\mu\nu}) + F(\Psi)\nabla^2\Psi + 16\pi\mathcal{L}_m(\Phi)] \quad (7)$$

and is clearly a function of three independent gravitational field variables: the connection, the metric and the dilaton field.

Note that although $\nabla_\mu\Psi = \partial_\mu\Psi$ because Ψ is a scalar, since metricity is not assumed, $\nabla^2\Psi$ above is given explicitly by $\nabla^\mu\partial_\mu\Psi$ or $g^{\mu\nu}\nabla_\nu\partial_\mu\Psi$. Clearly in an *a priori* metric theory both the third and fourth terms above are identically zero, while the fifth merely adds a total divergence combined with a redefinition of the $A(\Psi)$ term. However, if we are to take the spirit of the Palatini variation seriously [9], i.e. assume potential non-metricity from the outset, then these terms must occur for completeness.

Upon investigating the dynamics resulting from a Palatini variation of the above action, we find the circumstances under which metric compatibility explicitly occurs in the general N -dimensional case. We find as well that the case $N = 2$ merits special attention; and we investigate the differing field and geometrical relationships which arise in this context. Finally, using conformal transformations, we examine the constraints required to establish an equivalence between the above "Palatini dynamics" and those instead derived from the more usual "Hilbert variational principle" - i.e. mandating *a priori* the equivalence of the connection with the Christoffel symbol and then varying solely with respect to the metric and the dilaton field.

2 N -Dimensional Dynamics

If we vary the action (7) with respect to the connection, metric and dilaton field respectively, we find the following field equations:

$$\left\{ \frac{-1}{\sqrt{-g}} \nabla_\lambda \left[D\sqrt{-g}g^{\mu\nu} \right] + \frac{1}{2\sqrt{-g}} \nabla_\rho \left[D\sqrt{-g} (g^{\mu\rho}\delta_\lambda^\nu + g^{\nu\rho}\delta_\lambda^\mu) \right] \right.$$

$$+(B + \frac{1}{2}C) [(\partial^\nu \Psi)\delta_\lambda^\mu + (\partial^\nu \Psi)\delta_\lambda^\nu] + (C - F)(\partial_\lambda \Psi)g^{\mu\nu} \} = 0 \quad (8)$$

$$\begin{aligned} 8\pi T_{\mu\nu} = & DG_{(\mu\nu)} + A \left[\partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} (\partial \Psi)^2 \right] - B(\partial_\mu \Psi) \frac{\nabla_\nu \sqrt{-g}}{\sqrt{-g}} - B(\partial_\nu \Psi) \frac{\nabla_\mu \sqrt{-g}}{\sqrt{-g}} \\ & - B(\partial^\eta \Psi) g_{\rho\mu} g_{\epsilon\nu} \nabla_\eta g^{\rho\epsilon} - \frac{1}{2} C g_{\mu\nu} (\partial_\rho \Psi) (\nabla_\eta g^{\eta\rho}) + F \left[\nabla_\mu (\partial_\nu \Psi) - \frac{1}{2} g_{\mu\nu} \nabla^2 \Psi \right] \\ & - \nabla_\rho [B \partial^\rho \Psi g_{\mu\nu}] - \frac{1}{2\sqrt{-g}} \nabla_\nu [C \sqrt{-g} (\partial_\mu \Psi)] - \frac{1}{2\sqrt{-g}} \nabla_\mu [C \sqrt{-g} (\partial_\nu \Psi)] \end{aligned} \quad (9)$$

$$\begin{aligned} \{ D'R + A'(\partial \Psi)^2 - 2B'(\partial^\rho \Psi) \frac{\nabla_\rho \sqrt{-g}}{\sqrt{-g}} + C'(\partial_\mu \Psi) \nabla_\nu g^{\mu\nu} + F'(\nabla^2 \Psi) \\ + \frac{\nabla_\mu}{\sqrt{-g}} \nabla_\nu [F \sqrt{-g} g^{\mu\nu}] - \frac{\nabla_\rho}{\sqrt{-g}} [2A \sqrt{-g} (\partial^\rho \Psi)] \\ + \frac{2\nabla_\mu}{\sqrt{-g}} \left[B \sqrt{-g} g^{\mu\nu} \left(\frac{\nabla_\nu \sqrt{-g}}{\sqrt{-g}} \right) \right] - \frac{\nabla_\mu}{\sqrt{-g}} [C \sqrt{-g} \nabla_\nu g^{\mu\nu}] \} = 0 \end{aligned} \quad (10)$$

where the explicit dependence of A, B, C, D and F on Ψ is suppressed for notational convenience; and A' , say, represents $\frac{\partial A}{\partial \Psi}$.

Consider next simplification of the connection equation (8). After contracting λ and μ we find

$$\frac{\nabla_\rho}{\sqrt{-g}} [D \sqrt{-g} g^{\rho\nu}] = (\partial^\nu \Psi) \left[\frac{2}{1-N} \right] \left[(N+1)B + \left(\frac{N+3}{2} \right) C - F \right] \quad (11)$$

which yields upon substitution back into (8)

$$(2-N)D \left(\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} \right) = \left[\frac{(\partial_\lambda \Psi)}{N-1} \right] [N(N-1)D' + 4B + (4-N^2+N)C + (N-2)(N+1)F], \quad (12)$$

after tracing over μ and ν .

Therefore, assuming $N \neq 2$, we find that (8) becomes

$$\nabla_\lambda g^{\mu\nu} = X(\partial_\lambda \Psi) g^{\mu\nu} + Y [(\partial^\mu \Psi) \delta_\lambda^\nu + (\partial^\nu \Psi) \delta_\lambda^\mu], \quad (13)$$

where (11) and (12) have been employed, and where

$$X(\Psi) = \left\{ \frac{2[(1-N)D' - 2B + (N-3)C + (2-N)F]}{D(N-2)(1-N)} \right\}, \quad (14)$$

and

$$Y(\Psi) = \left[\frac{2(B+C) - F}{D(1-N)} \right]. \quad (15)$$

By permuting (13) we can obtain an explicit solution for the connection

$$\Gamma_{\mu\nu}^{\eta} = \left\{ \begin{matrix} \eta \\ \mu \quad \nu \end{matrix} \right\} + \left(Y - \frac{1}{2}X \right) (\partial^{\eta}\Psi)g_{\mu\nu} + \frac{1}{2}X \left[(\partial_{\mu}\Psi)\delta_{\nu}^{\eta} + (\partial_{\nu}\Psi)\delta_{\mu}^{\eta} \right] \quad (16)$$

in terms of the metric and dilaton.

From the form of $\Gamma_{\mu\nu}^{\eta}$ above, we see that it is still symmetric in the lower two indices. Hence $R_{\alpha\beta}$ and $G_{\alpha\beta}$ remain symmetric tensors and we can replace $G_{(\alpha\beta)}$ in (3) by just $G_{\alpha\beta}$. Furthermore, we can use the explicit form of the connection as given above to obtain a general expression for $G_{\mu\nu}$ in terms of the Christoffel symbols and the dilaton factors. From (3) we have

$$\begin{aligned} R_{\mu\nu} &= R_{\mu\nu}(\{\}) + \left[(Y' - \frac{1}{2}X') - (Y - \frac{1}{2}X) \left[\left(\frac{N-2}{2} \right) X - NY \right] \right] (\partial\Psi)^2 g_{\mu\nu} \\ &\quad + \left[\left[\frac{2-N}{2}X' - Y' \right] + \frac{4-N}{4}X^2 - Y^2 \right] \partial_{\mu}\Psi \partial_{\nu}\Psi \\ &\quad + \left[\frac{2-N}{2}X - Y \right] \nabla_{\mu}\partial_{\nu}\Psi + \left[Y - \frac{1}{2}X \right] (\nabla^2\Psi)g_{\mu\nu} \end{aligned} \quad (17)$$

which upon insertion into (9) yields

$$8\pi T_{\mu\nu} = DG_{\mu\nu}(\{\}) + \alpha(\partial\Psi)^2 g_{\mu\nu} + \beta(\partial_{\mu}\Psi)(\partial_{\nu}\Psi) - D'[\nabla_{\mu}(\partial_{\nu}\Psi) - (\nabla^2\Psi)g_{\mu\nu}], \quad (18)$$

where

$$\begin{aligned} \alpha(\Psi) &= D'' + \frac{1}{2}F' - \frac{1}{2}A + \frac{1}{2}D'[(3-N)X + 2(N-1)Y] \\ &\quad + \frac{1}{4}(1-N)D \left[\left(\frac{N-2}{2} \right) X^2 - 2Y^2 - (N-2)XY \right] \end{aligned} \quad (19)$$

$$\beta(\Psi) = A - F' - D'' - D'X + \frac{1}{2}(N-1)D \left[\left(\frac{N-2}{2} \right) X^2 - 2Y^2 - (N-2)XY \right]. \quad (20)$$

and where (12) has been used.

A similar substitution transforms (10) into

$$D'R(\{\}) + \Pi(\partial\Psi)^2 + \Lambda(\nabla^2\Psi) = 0, \quad (21)$$

where

$$\begin{aligned} \Pi(\Psi) &= F'' - A' + A[(N-2)X - 2NY] + 2F' \left[\left(\frac{2-N}{2} \right) X + NY \right] \\ &\quad + \frac{1}{2}(1-N)D' \left[\left(\frac{N-2}{2} \right) X^2 - 2Y^2 - (N-2)XY \right] \\ &\quad + (1-N)D \left[\left(\frac{N-2}{2} \right) XX' - 2YY' - \left(\frac{N-2}{2} \right) [XY' + YX'] \right], \end{aligned} \quad (22)$$

and

$$\Lambda(\Psi) = 2F' - 2A + (1 - N)D \left[\left(\frac{N-2}{2} \right) X^2 - 2Y^2 - (N-2)XY \right]. \quad (23)$$

It will later prove convenient to re-express (18) and (21) via (16) directly in terms of \mathcal{D}_μ , defined as the covariant derivative, ∇_μ , with the connection equivalent to the Christoffel symbol. In this way, then, we find that (18) becomes:

$$8\pi T_{\mu\nu} = DG_{\mu\nu}(\{\}) + \tilde{\alpha}(\partial\Psi)^2 g_{\mu\nu} + \tilde{\beta}(\partial_\mu\Psi)(\partial_\nu\Psi) - D' \left[D_\mu(\partial_\nu\Psi) - (\mathcal{D}^2\Psi)g_{\mu\nu} \right], \quad (24)$$

where

$$\tilde{\alpha} := \alpha - \frac{1}{2}D'[(3-N)X + 2(N-1)Y], \quad (25)$$

$$\tilde{\beta} := \beta - D'X, \quad (26)$$

and where $\mathcal{D}^2\Psi$ is defined in the usual way as

$$\mathcal{D}^2\Psi := g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\Psi. \quad (27)$$

Meanwhile (21) becomes

$$D'R(\{\}) + \tilde{\Pi}(\partial\Psi)^2 + \Lambda(\mathcal{D}^2\Psi) = 0, \quad (28)$$

where

$$\begin{aligned} \tilde{\Pi} = & F'' - A' + \frac{1}{2}(1-N)D' \left[\left(\frac{N-2}{2} \right) X^2 - 2Y^2 - (N-2)XY \right] \\ & + (1-N)D \left[\left(\frac{N-2}{2} \right) XX' - 2YY' - \left(\frac{N-2}{2} \right) [XY' + YX'] \right. \\ & \left. + \left\{ \frac{2-N}{2} X^2 + 2Y^2 + (N-2)XY \right\} \left\{ \frac{2-N}{2} X + NY \right\} \right] \end{aligned} \quad (29)$$

As a way of checking these dynamical equations, consider the behaviour of the matter action under an infinitesimal coordinate transformation [2]. For coordinate invariance of the matter action the condition

$$\partial_\nu \left[\sqrt{-g} T_\lambda^\nu \right] - \frac{1}{2} (\partial_\lambda g_{\mu\nu}) \sqrt{-g} T^{\mu\nu} = 0. \quad (30)$$

must hold. In the metrically compatible case, this leads directly to the covariant conservation of the stress energy tensor, $\nabla^\eta T_{\eta\lambda} = 0$. However in the more general dilatonic case, with the connection determined by (16) above, we have instead the condition

$$\nabla^\eta T_{\eta\lambda} = -\frac{1}{2}TX(\partial_\lambda\Psi) + T_{\eta\lambda} \left[\left(\frac{N-2}{2} \right) X - (N+1)Y \right] (\partial^\eta\Psi) \equiv \mathcal{W}_\lambda, \quad (31)$$

where T represents the trace of the stress-energy tensor, $g^{\mu\nu}T_{\mu\nu}$. Explicitly evaluating \mathcal{W}_λ using (18) yields

$$\begin{aligned} \mathcal{W}_\lambda = & \left(\frac{1}{8\pi} \right) \{ DR_{\eta\lambda}(\{\}) (\partial^\eta\Psi) [(\frac{N-2}{2})X - Y(N+1)] \\ & - D'(\partial^\eta\Psi)(\nabla_\eta(\partial_\lambda\Psi)) [(\frac{N-2}{2})X - Y(N+1)] \\ & + (\partial_\lambda\Psi) [\frac{1}{2}DR(\{\})Y(N+1) - D'(\frac{1}{2}X + (N+1)Y)(\mathcal{D}^2\Psi) \\ & - \{X(\frac{1}{2}\beta(3-N) + \alpha) + (N+1)Y(\alpha + \beta)\}(\partial\Psi)^2] \} \end{aligned} \quad (32)$$

Alternatively if we compute $\nabla^\eta T_{\eta\lambda}$ directly, by operating on $T_{\eta\lambda}$ as given in (18) above with the operator ∇^η we obtain

$$\nabla^\eta T_{\eta\lambda} = \mathcal{W}_\lambda - \frac{1}{2}(\partial_\lambda\Psi) \left[D'R(\{\}) + \tilde{\Pi}(\partial\Psi)^2 + \Lambda(\mathcal{D}^2\Psi) \right], \quad (33)$$

where we have used (16), where \mathcal{W}_λ is given by (32).

Hence the covariant conservation of the stress-energy is satisfied whenever the dilaton field equation (21), is satisfied as well. The conservation law (31) generalizes to the Palatini formalism that found in ref. [4] for dilaton gravity theories.

3 $N = 2$ Dynamics

We can see from the form of equation (12) that for $N = 2$ the approach given above will break down: we will no longer be able to find an explicit expression for $\left(\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} \right)$, and hence eventually $\nabla_\lambda g^{\mu\nu}$ in terms of functions of the dilaton field and its derivative. Instead, for $N = 2$, we are merely left with an added constraint:

$$D' + 2B + C = 0. \quad (34)$$

Note that if (34) does not hold then from (12) the dilaton must be constant $\Psi = \Psi_0$. The field equations (8), (9) then reduce to

$$\frac{-1}{\sqrt{-g}} \nabla_\lambda \left[D_0 \sqrt{-g} g^{\mu\nu} \right] = 0 \quad (35)$$

and

$$8\pi T_{\mu\nu} = D_0 G_{(\mu\nu)}(\Gamma) \quad (36)$$

where $D_0 = D(\Psi_0)$ is constant. This situation was previously investigated in ref [10]. Although it appears to yield non-trivial dynamics, this does not occur because eq. (35) is invariant under the transformation

$$\Gamma_{\mu\nu}^\eta \Rightarrow \hat{\Gamma}_{\mu\nu}^\eta = \Gamma_{\mu\nu}^\eta + A_\mu \delta_\nu^\eta + A_\nu \delta_\mu^\eta - g_{\mu\nu} A^\eta, \quad (37)$$

where A_λ is an arbitrary vector field. From this it may be shown [10] that the general solution to (35) is

$$\Gamma_{\mu\nu}^\eta = \left\{ \begin{matrix} \eta \\ \mu \quad \nu \end{matrix} \right\} + A_\mu \delta_\nu^\eta + A_\nu \delta_\mu^\eta - g_{\mu\nu} A^\eta \quad (38)$$

where A_μ is undetermined. Insertion of this into the right hand side of (36) yields $G_{(\mu\nu)}(\Gamma) = 0$. Hence the theory is either inconsistent (if $T_{\mu\nu} \neq 0$) or trivial (if $T_{\mu\nu} = 0$).

For Ψ not constant we can understand the constraint (34) in the following way. For $N = 2$ the associated action (7) is invariant under the transformation (36) provided the constraint (34) is valid. Since A_λ is arbitrary, we can choose it in such a way as to achieve explicit dynamical equations for $N = 2$. Since under (37)

$$\frac{\hat{\nabla}_\lambda \sqrt{-g}}{\sqrt{-g}} = \frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} - 2A_\lambda. \quad (39)$$

we chose

$$A_\lambda = \frac{1}{2} \left(\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} \right) \quad (40)$$

so that

$$\nabla_\lambda g^{\mu\nu} = Y [(\partial^\mu \Psi) \delta_\lambda^\nu + (\partial^\nu \Psi) \delta_\lambda^\mu - (\partial_\lambda \Psi) g^{\mu\nu}] \quad (41)$$

and

$$\Gamma_{\alpha\beta}^\eta = \left\{ \begin{matrix} \eta \\ \alpha \quad \beta \end{matrix} \right\} - \frac{1}{2} Y [(\partial_\alpha \Psi) \delta_\beta^\eta + (\partial_\beta \Psi) \delta_\alpha^\eta - 3g_{\alpha\beta} (\partial^\eta \Psi)] \quad (42)$$

where the hat notation has been dropped and $B(\Psi)$ has been eliminated using (34). Inserting these equations into the field equations (9) and (10), together with (42) yields

$$\begin{aligned} 8\pi T_{\mu\nu} = & \left[\frac{1}{2} D(Y' + Y^2) + \frac{1}{2} D'Y(4 - 3N) \frac{1}{2} A - B' - Y(C + 2B) \right] (\partial\Psi)^2 g_{\mu\nu} \\ & + [A - C' - D(Y' + Y^2) - D'Y] (\partial_\mu \Psi) (\partial_\nu \Psi) - D'[\mathcal{D}_\mu (\partial_\nu \Psi) - (\mathcal{D}^2 \Psi) g_{\mu\nu}] \end{aligned} \quad (43)$$

and

$$\begin{aligned}
& \{F'' - A' + 2Y'(F - C) + Y[(3N - 6)A + (2 - 3N)F'] \\
& + Y^2[7D' + 6(F - C) - 3N(F - C + D')]\}(\partial\Psi)^2 \\
& + D'R(\{\}) + 2[F' + Y(F - C + D') - A](\mathcal{D}^2\Psi) = 0,
\end{aligned} \tag{44}$$

That is,

$$D'R(\{\}) + \hat{\Pi}(\partial\Psi)^2 + \hat{\Lambda}(\mathcal{D}^2\Psi) = 0, \tag{45}$$

with the obvious definitions for $\hat{\Pi}$ and $\hat{\Lambda}$ in accordance with (44) above. The conservation law (31) holds where now

$$\mathcal{W}_\lambda := -\frac{1}{2}TX(\partial_\lambda\Psi) - 3T_{\eta\lambda}Y(\partial^\eta\Psi) \quad . \tag{46}$$

However operating with ∇^η on both sides of (43) leads to

$$\nabla^\eta T_{\eta\lambda} = \mathcal{W}_\lambda - \frac{1}{2}(\partial_\lambda\Psi) \left[D'R(\{\}) + \hat{\Pi}(\partial\Psi)^2 + \hat{\Lambda}(\mathcal{D}^2\Psi) \right] \tag{47}$$

as with the $N > 2$ case. Once again we see that conservation of stress energy is consistently satisfied provided the dilaton field equation is satisfied as well.

4 Analysis

We turn now to a comparison of the Palatini method to the “Hilbert variational method” - i.e. the method of varying only with respect to the metric and the dilaton field, assuming the metric compatibility condition (6) is satisfied. In our formalism, the E-H action (2) is equivalent to a special case of the action (7) with $D = 1$; $A = B = C = F = 0$, which in turn implies, via (14) and (15), that $X = Y = 0$, and hence the constraint (6), $\Gamma = \{\}$.

Can these ideas be generalized? Clearly from (16), we will recover explicit metricity if $X = Y = 0$. From (14) and (15), this immediately implies (34).

Yet, somewhat surprisingly, this is not the only case where the dynamics deduced from a Palatini variation agree with those deduced from a Hilbert variation. Lindström [12, 13] showed, when examining Brans-Dicke-type theories under a Palatini variational principle, that both the Palatini and Hilbert variations yield identical dynamics, the only

(nominal) difference occurring in the value of the dimensionless coupling constant ω . More specifically, under a Palatini variation of the general action

$$S = \int d^4x \sqrt{-g} \left[R\psi^\alpha - (\omega\psi^{\alpha-2}) (\partial\psi)^2 + \psi^\beta 16\pi \mathcal{L}_m \right], \quad (48)$$

he showed that the Palatini induced dynamics are equivalent to the Hilbert ones, with a rescaling of the (dimensionless) coupling constant:

$$\omega \rightarrow \hat{\omega} = \omega - \frac{3\alpha^2}{2}. \quad (49)$$

The justification for this equivalence lies in the fact that for this particular action the form of the connection constraint (16) is

$$\Gamma_{\mu\nu}^\eta = \left\{ \begin{matrix} \eta \\ \mu \quad \nu \end{matrix} \right\} + \left(\frac{\alpha}{2\psi} \right) \left[(\partial_\mu \psi) \delta_\nu^\eta + (\partial_\nu \psi) \delta_\mu^\eta - g_{\mu\nu} (\partial^\eta \psi) \right], \quad (50)$$

which we recognize as that of an induced metrically-compatible connection after a conformal transformation

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \psi^\alpha g_{\mu\nu}; \psi \rightarrow \hat{\psi} = \psi, \quad (51)$$

expressed in terms of the “old” metric $g_{\mu\nu}$. Therefore if we apply the following conformal transformation, henceforth known as a “Palatini Transformation” (owing to its explicit mention of the connection)

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \psi^\alpha g_{\mu\nu}; \psi \rightarrow \hat{\psi} = \psi; \Gamma_{\rho\tau}^\eta \rightarrow \hat{\Gamma}_{\rho\tau}^\eta = \Gamma_{\rho\tau}^\eta, \quad (52)$$

to the action (48), we find

$$\hat{S} = \int d^4x \sqrt{-\hat{g}} \left[\hat{R} - \left(\frac{\omega}{\psi^2} \right) (\hat{\partial}\psi)^2 + \psi^{\beta-2\alpha} \mathcal{L}_m \right], \quad (53)$$

which can now be regarded as an Einstein-Hilbert action coupled to matter. The Palatini variation acting upon the transformed action (53) gives simply the Christoffel relation (6), and thus for this action (53) both the Hilbert and Palatini variational methods lead to identical results for the connection. To obtain the dynamics of (48), then, one can apply the Palatini variational principle to (53) and then subject the results to the “inverse Palatini transformation”:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \psi^{-\alpha} g_{\mu\nu}; \psi \rightarrow \hat{\psi} = \psi; \Gamma_{\rho\tau}^\eta \rightarrow \hat{\Gamma}_{\rho\tau}^\eta = \Gamma_{\rho\tau}^\eta \quad (54)$$

Similarly, if one starts again from the action (53) and assumes, under a "Hilbert perspective" that the connection is always equivalent to the Christoffel symbol, it can be easily shown that (53) can be transformed into

$$S = \int d^4x \sqrt{-g} \left[R\psi^\alpha - (\hat{\omega}\psi^{\alpha-2}) (\partial\psi)^2 + \psi^\beta 16\pi \mathcal{L}_m \right], \quad (55)$$

i.e. (48) with $\hat{\omega}$, as given by (49), instead of ω . This treatment by Lindström linking the Palatini and Hilbert approaches suggests that we look at an analogous form of our Palatini connection equation, i.e. when the connection in (16) has the form of a Christoffel symbol after an associated conformal transformation. Inspection of (16) indicates this happens if $Y = 0 \Leftrightarrow 2B + 2C - F = 0$. The connection equation then becomes

$$\Gamma_{\mu\nu}^\eta = \left\{ \begin{matrix} \eta \\ \mu \quad \nu \end{matrix} \right\} + \frac{1}{2} X \left[(\partial_\mu \Psi) \delta_\nu^\eta + (\partial_\nu \Psi) \delta_\mu^\eta - (\partial^\eta \Psi) g_{\mu\nu} \right] \quad (56)$$

which is identical to the form of the induced Christoffel connection after a conformal transformation of the metric:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (57)$$

where

$$\partial_\epsilon (\ln \Omega) = \frac{X}{2} \partial_\epsilon \Psi. \quad (58)$$

Therefore, if we apply the following Palatini transformation to the action (7)³:

$$\Gamma_{\rho\xi}^\eta \rightarrow \hat{\Gamma}_{\rho\xi}^\eta = \Gamma_{\rho\xi}^\eta; \Psi \rightarrow \hat{\Psi} = \Psi; g_{\mu\nu} \rightarrow \Omega^2 \hat{g}_{\mu\nu} \quad (59)$$

we find the transformed action

$$\begin{aligned} \hat{S} = \int d^N x \sqrt{-\hat{g}} \{ & \Omega^{2-N} [D\hat{R} + A(\hat{\partial}\Psi)^2 + B(\hat{\partial}^\eta \Psi) \hat{g}_{\mu\nu} \hat{\nabla}_\eta \hat{g}^{\mu\nu} + C(\partial_\mu \Psi) (\hat{\nabla}_\nu \hat{g}^{\mu\nu}) \\ & + F \hat{\nabla}^\eta (\partial_\eta \Psi)] + 16\pi \hat{\mathcal{L}}_m + 2(NB + C) \Omega^{1-N} (\partial_\mu \Psi) (\partial_\nu \Omega) \hat{g}^{\mu\nu} \} \end{aligned} \quad (60)$$

Applying the Palatini principle to this action together with the above constraints, $Y = 0 \Leftrightarrow 2B + 2C - F = 0$ and eq. (58) results in the Hilbert constraint (6) for the connection equation, i.e. $\Gamma = \{\}$. This may be seen as follows. Variation of the above action with respect to $\Gamma_{\mu\nu}^\lambda$ yields after some simplification the equation

$$\Omega^{2-N} \nabla_\alpha \psi g^{\mu\nu} [C - F - D'] - \frac{D}{\sqrt{-g}} \nabla_\alpha [\Omega^{2-N} \sqrt{-g} g^{\mu\nu}] = 0 \quad (61)$$

³recall that here the Ricci tensor, $R_{\mu\nu}$ is expressed *solely* as a function of the connection and is hence unchanged by the following transformation

where the constraint $Y = 0$ has been imposed. Note that for $Y = 0$

$$\frac{X}{2} = \left[\frac{D' - C + F}{D(N - 2)} \right] \quad (62)$$

which implies from (61)

$$\frac{1}{\sqrt{-g}} \nabla_\alpha [\sqrt{-g} g^{\mu\nu}] = 0 \quad (63)$$

where the above constraint (58) has been employed. For $N \neq 2$ it is straightforward to show that (63) implies that metricity holds. The $N = 2$ case follows from the same connection invariance arguments given in the previous section.

Unfortunately, the above analysis alone is not very helpful in answering the question of when the Hilbert and Palatini approach differ in regards to their *physical* content. Lindström was working with a particular action which had the unique property of being, after conformally transformed via the associated Palatini transformation and "re-transformed" via the associated Hilbert transformation, equivalent to the original action with the sole exception that $\omega \rightarrow \hat{\omega} = \omega - \frac{3(\alpha)^2}{2}$.

In general this is definitely not the case, i.e. one cannot generally say that the inverse Hilbert transformation:

$$\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Omega^{-2} \hat{g}_{\mu\nu}; \hat{\Psi} \rightarrow \Psi = \hat{\Psi} \quad (64)$$

applied to (53) will yield some other Hilbert action with merely a change in some dimensionless constant. Nonetheless, one can say something. To do this, we again regard our Palatini dynamical equations expressed explicitly in terms of the Christoffel symbol, i.e. (24) and (28), but this time subject to the explicit constraint $Y = 0$. Therefore our Palatini dynamical equations (24),(28) for $Y = 0$ now become:

$$\begin{aligned} 8\pi T_{\mu\nu} = & DG_{\mu\nu}(\{\}) + \left[D'' + \frac{1}{2} (F' - A + \tilde{Q}) \right] (\partial\Psi)^2 g_{\mu\nu} \\ & - [D'' + (F' - A + \tilde{Q})] (\partial_\mu \Psi) (\partial_\nu \Psi) - D' [\mathcal{D}_\mu (\partial_\nu \Psi) - \mathcal{D}^2 \Psi], \end{aligned} \quad (65)$$

and

$$D'R(\{\}) + [(F' - A + \tilde{Q})'] (\partial\Psi)^2 + 2 [F' - A + \tilde{Q}] \mathcal{D}^2 \Psi = 0, \quad (66)$$

where

$$\tilde{Q} := (N - 1) \frac{2 - N}{4} DX^2 = \frac{(N - 1)(D' - C + F)^2}{D(2 - N)}. \quad (67)$$

Meanwhile a Hilbert variation of our original action (7) yields:

$$8\pi T_{\mu\nu} = DG_{\mu\nu} + \left[D'' + \frac{1}{2}(F' - A) \right] (\partial\Psi)^2 g_{\mu\nu} - [D'' + (F' - A)] (\partial_\mu\Psi)(\partial_\nu\Psi) - D'[\mathcal{D}_\mu(\partial_\nu\Psi) - \mathcal{D}^2\Psi], \quad (68)$$

and

$$D'R(\{\}) + [(F' - A)'] (\partial\Psi)^2 + 2[F' - A] \mathcal{D}^2\Psi = 0, \quad (69)$$

Therefore, provided that \tilde{Q} is proportional to $(F' - A)$, the Palatini dynamics will merely result in a rescaling of the dimensionless function $F' - A$ and so will be equivalent to the Hilbert dynamics, as in the above special case examined by Lindström. Note that \tilde{Q} is a quadratic function of D' , C and F , and so this allows for a more general set of relationships between the dilatonic functions.

In order to illustrate the point, let us take, for example, the following case (for $a, b, c, d, f \in \mathcal{R}$):

$$A = ae^{k\Psi}; D = de^{k\Psi}; C = ce^{k\Psi}; F = fe^{k\Psi} \quad (70)$$

and hence, by $Y = 0$,

$$B = \frac{f - 2c}{2} e^{k\Psi} \quad (71)$$

. Therefore, we now have:

$$\tilde{Q} = \frac{(N - 1)(kd - c + f)^2}{(2 - N)d} e^{k\Psi}, \quad (72)$$

and clearly we now have a case where the Palatini dynamics are physically equivalent to the Hilbert dynamics, the only difference being, as before for the Brans-Dicke-like theories, that the (physically irrelevant) dimensionless constant, $\omega := kf - a$, now becomes:

$$\hat{\omega} = \omega + \frac{(N - 1)(kd - c + f)^2}{(2 - N)d} \quad (73)$$

5 Conclusions

We have examined the explicit dynamics of a general second-order, N -dimensional, torsion-free dilaton gravity action under the Palatini variational principle and checked these dynamics by considering the invariance of the matter action under simple coordinate transformations. Unlike general relativity, derived from the standard Einstein-Hilbert action, a Palatini variation of the action does not generally lead to equivalent dynamics to that of a Hilbert variation (i.e. the dynamics obtained by assuming *a priori* that the connection in the action is that of the Christoffel symbol). Instead the dynamics are only identical provided that $D' + 2B + C = 0$. Furthermore, when both $2B + 2C - F = 0$, and $\tilde{Q} = \frac{(N-1)}{D(N-2)}(D' - C + F)^2$ is proportional to $(F' - A)$, then the two approaches merely differ by a physically irrelevant constant.

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